6. D. F. Anderson, S. Chapman, and W. W. Smith, *Some factorization properties of Krull domains with infinite cyclic divisor class group*, J. Pure and Appl. Algebra **96** (1994), 97-112.
7. D. F. Anderson, S. Chapman, F. Inman, and W.W. Smith, *Factorization in* $K[X^2, X^3]$, Arch. Math. **61** (1993), 521-528.
8. D. F. Anderson and S. Jenkens, *Factorization in* $K[X^n, X^{n+1}, \ldots, X^{2n-1}]$, Comm. Algebra **234** (1995), 2561-2576.
9. D. F. Anderson and P. Pruis, *Length functions on integral domains*, Proc. Amer. Math. Soc. **113** (1991), 933-937.
10. A. Brauer and J.E. Schockley, *On a problem of Frobenius*, J. Reine Angew. Math. **211** (1962), 215-220.
11. S. Chapman, *On the Davenport constant, the cross number, and their applications in factorization theory*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, volume 171, 1995, chapter 14, pp. 167-190.
12. S. Chapman and W. W. Smith, *An analysis using the Zaks-Skula constant of element factorizations in Dedekind domains*, J. Algebra **159** (1993), 176-190.
13. S. Chapman and W. W. Smith, *Factorization in Dedekind domains with finite class group*, Israel J. Math. **71** (1990), 65-95.
14. R. Fröberg, G. Gottlieb, and R. Häggkvist, *On numerical semigroups*, Semigroup Forum **35** (1987), 63-83.
15. A. Geroldinger, *Über nicht-eindeutige Zerlegungen in irreduzible Elements*, Math. Z. **197** (1988), 505-529.
16. A. Geroldinger and R. Schneider, *On Davenport's constant*, J. Combin. Theory Ser. **A 61** (1992), 147-152.
17. R. Gilmer, *Commutative Semigroup Rings*, Chicago Lectures in Mathematics, Univ. of Chicago Press, Chicago, 1984.
18. H. Kornblum, *Über die Primfunktionen in einer arithmetischen Progession*, Math. Z. **5** (1919), 100-111.
19. M. Roitman, *Polynomial extensions of atomic domains*, J. Pure Appl. Algebra **87** (1993), 187-199.
20. E.S. Selmer, *On a linear diophantine problem of Frobenius*, J. Reine Angew. Math. **293/294** (1977), 1-17.
21. J. L. Steffan, *Longuers des décompositions en produits d'éléments irréductibles dans un anneau de Dedekind*, J. Algebra **102** (1986), 229-236.
22. R. J. Valenza, *Elasticity of factorizations in number fields*, J. Number Theory **36** (1990), 212-218.
23. A. Zaks, *Half-factorial domains*, Israel J. Math. **37** (1980), 281-302.

Pseudo-Valuation Rings

AYMAN BADAWI French House, 710 West 21st Street, Austin, Texas 78705 USA

DAVID F. ANDERSON and DAVID E. DOBBS Department of Mathematics, The University of Tennessee, Knoxville, Tennessee 37996-1300 USA

1. INTRODUCTION

Throughout this paper, all rings are commutative with identity and if R is a ring, then $Z(R)$ denotes the set of zerodivisors of R. Our main purpose is to generalize the study of pseudo-valuation domains (as introduced in [8]) to the context of arbitrary rings (with $Z(R)$ possibly nonzero). Recall from [8] that an integral domain R, with quotient field K, is called a *pseudo-valuation domain* (PVD) in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$; equivalently, in case R has a (uniquely determined) valuation overring V such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ as sets; equivalently (by [3, Proposition 2.6]), in case R is a pullback of the form $V \times_F k$, where V is a valuation domain with residue field F and k is a subfield of F; equivalently, in case (R, M) is a quasilocal domain such that $(M : M) = \{ x \in K \mid xM \subset M \}$ is a valuation domain with maximal ideal M. Additional characterizations of PVDs are known, for instance in terms of comparability of fractional ideals (cf. [1], [2], [4], [5]).

The first part of this paper concerns the extent to which the above equivalences carry over to the more general ring-theoretic context, as we study pseudo-valuation rings (PVRs, defined in Section 2). By analogy with known results on PVDs, we also consider the stability of the class of PVRs with respect to homomorphic images and localization. The second part of this paper concerns stability of the class of PVRs under passage to overrings; and the related issue of whether R' = V, where (R, M) is a pseudo-valuation ring with integral closure R' and (M : M) = V. This work is motivated by the result (cf. [9, Proposition 2.7], [6, Proposition 4.2]) that if R is a PVD with canonically associated valuation domain V, then each overring of R is a PVD if and only if R' = V.

Prior study of the "strongly prime ideal" concept and its generalizations has focused on the case of integral domains, as in [8], [7], [2], [4], and [5]. For this reason, we emphasize here the new role of Z(R), while following methods familiar from the domain-theoretic context as much as possible. Most of our departures from those methods arise because not all the relevant results on PVDs generalize without qualification to PVRs. Any unexplained material is standard, as in [10].

2. RESULTS

A prime ideal P of a ring R is said to be *strongly prime* if aP and bR are comparable for all a, b ∈ R. If R is an integral domain, this is equivalent to the definition in the introduction (cf. [1, Proposition 3.1], [2, Proposition 4.2], [5, Proposition 3]). We shall say that a ring R is a *pseudo-valuation ring (PVR)* if each prime ideal of R is strongly prime. Examples of PVRs include PVDs, in particular, valuation domains, and by Corollary 3, any homomorphic image of a valuation domain. Lemma 1 and Theorem 2 are the analogues for rings with zerodivisors of [8, Corollary 1.3 and Theorem 1.4].

Lemma 1. (a) Let I be an ideal of a ring R and P a strongly prime ideal of R. Then I and P are comparable.

(b) Any PVR is quasilocal.

Proof. (a) Suppose that I is not contained in P. Then for b ∈ I - P and a = 1, bR is not contained in P = aP, and so P ⊂ bR ⊂ I.

(b) This follows easily from (a). □

Theorem 2. A quasilocal ring R with maximal ideal M is a PVR if and only if M is strongly prime.

Proof. By definition, M is strongly prime if R is a PVR. Conversely, suppose that M is strongly prime. We must show that each nonmaximal prime ideal P of R is strongly prime. Let a, b ∈ R. We show that aP and bR are comparable. Since M is strongly prime, aM and bR are comparable. If aM ⊂ bR, then aP ⊂ aM ⊂ bR. Thus we may assume that bR is properly contained in aM, and hence b = am for some m ∈ M. If m ∈ P, then b = am ∈ aP, and hence bR ⊂ aP. Thus we may assume m ∉ P. We show that P ⊂ mM. Let x ∈ P. Then xR and mM are comparable. If mM ⊂ xR ⊂ P, then either m ∈ P or M ⊂ P, a contradiction. Thus xR ⊂ mM for all x ∈ P, and hence P ⊂ mM. Thus aP ⊂ amM = bM ⊂ bR. □

Corollary 3. Any homomorphic image of a PVR is a PVR. □

Recall that a ring R is a *chained ring* if its ideals are linearly ordered by inclusion (equivalently, its principal ideals are linearly ordered by inclusion). Our next corollary is the zerodivisor analogue of the result that any valuation domain is a PVD [8, Proposition 1.1].

Corollary 4. Any chained ring R is a PVR.

Proof. Let R be a chained ring with maximal ideal M and a, b ∈ R. Since R is a chained ring, the ideals aM and bR are comparable. Thus M is strongly prime, and so R is a PVR by Theorem 2. □

We next give several equivalent "comparability" conditions for a (not necessarily quasilocal) ring R to be a PVR (cf. [5, Proposition 3] and [8, Theorem 1.4]).

Theorem 5. Let N be the set of all nonunit elements of a ring R. Then the following conditions are equivalent:

(1) R is a PVR;

(2) For all $a, b \in R$, either $a | b$ or $b | an$ for all $n \in N$;

(3) For all ideals I and J of R, either $I \subset J$ or $JL \subset I$ for every proper ideal L of R;

(4) For all $a, b \in R$, either $a | b$ or $aN \subset bN$.

Proof. (1) $\Rightarrow$ (2): This follows easily since if R is a PVR, then R is quasilocal with strongly prime maximal ideal M ($= N$). (2) $\Rightarrow$ (3): Suppose that $I \not\subset J$. If $i \in I - J$ and $j \in J$, then by (2), $i | jr$ for all $r \in L \subset N$. Thus $JL \subset I$. (3) $\Rightarrow$ (4): Let $I = bR$ and $J = aR$, and apply (3). (4) $\Rightarrow$ (1): We first observe that R is quasilocal. If not, R has distinct maximal ideals P and Q. Choose $a \in P - Q$ and $b \in Q - P$. If $a | b$, then $b \in P$, a contradiction. If $aN \subset bN$, then $a^2 \in bN \subset Q$, and hence $a \in Q$, also a contradiction. Hence by (4), R is quasilocal with maximal ideal M, so $N = M$. Let $a, b \in R$. Suppose $b = ar$ with $r \in R$. Then $bR \subset aM$ if $r \in M$; otherwise, $a = r^{-1}b$ and $aM \subset bR$. If $a \nmid b$, then by (4), $aM \subset bM \subset bR$. Hence M is strongly prime, and so R is a PVR by Theorem 2. □

The converse of Corollary 4 is false. The easiest example is any nonvaluation PVD. A nondomain example is given in Example 10(a). Our next result shows how to construct a PVR as a pullback of a chained ring.

Theorem 6. Let V be a chained ring with maximal ideal M, $F = V/M$ its residue field, $\varphi : V \longrightarrow F$ the canonical epimorphism, k a subfield of F, and $R = \varphi^{-1}(k)$. Then the pullback $R = V \times_F k$ is a PVR.

Proof. First, note by standard pullback lore that R is quasilocal with maximal ideal M. Let $a, b \in R$. We show that aM and bR are comparable. Since V is a chained ring, either $a | b$ or $b | a$ in V; say $b = ra$ for some $r \in V$. If $r \in M$, then $b \in aM$, and hence $bR \subset aM$. So we may assume that r is a unit of V. Thus $a = r^{-1}b$ with $r^{-1} \in V$. For $m \in M$, $am = b(r^{-1}m) \in bM \subset bR$ since M is an ideal of V. Thus $aM \subset bR$. The $b | a$ case is similar. Hence R is a PVR by Theorem 2. □

In Corollary 9, we will give a partial converse to Theorem 6. Recall from [3, Theorem 3.10] that if R is a proper subring of a ring T, then $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ if and only if $\mathrm{Max}(R)$ is comparable to $\mathrm{Max}(T)$, and, in this case, R (and hence T) is quasilocal. In the spirit of [3], we have the next result.

Theorem 7. Let T be a quasilocal ring with maximal ideal M and R a subring of T with maximal ideal M (thus $\mathrm{Spec}(R) = \mathrm{Spec}(T)$). Then R is a PVR if and only if T is a PVR.

Proof. First suppose that R is a PVR. Let $a, b \in T$. We may assume that $a, b \in M$. Then aM and bR are comparable since R is a PVR. Then $aM \subset bR$ implies $aM \subset bR \subset bT$; so we may assume that $bR \subset aM$. But then $bT \subset aMT = aM$, and hence T is a PVR by Theorem 2.

Conversely, suppose that T is a PVR. Let $a, b \in R$. Again, we may assume that $a, b \in M$. Then aM and bT are comparable since T is a PVR. If $bT \subset aM$, then $bR \subset aM$; so we may assume that $aM \subset bT$. If aM is not contained in bR, then $am = bt$ for some $m \in M$ and $t \in T - R$. Hence t is a unit of T; so $b = a(mt^{-1}) \in aM$, and thus $bR \subset aM$. Hence R is a PVR by Theorem 2. □

Henceforth, let S denote the set of non-zerodivisors of a ring R and R_S the total quotient ring of R. If R is a quasilocal ring with maximal ideal M, we define $(M : M) = \{ x \in R_S \mid xM \subset M \}$, the largest overring of R (in R_S) in which M is an ideal. Let R be a quasilocal ring (for instance, a PVR) with maximal ideal M. If $M = Z(R)$, then $R_S = R$ and, in particular, $(M : M) = R$. If M contains a non-zerodivisor, then this observation may be strengthened as in the following result.

Theorem 8. Let R be a PVR whose maximal ideal M contains a non-zerodivisor. Then $V = (M : M)$ is a chained ring with maximal ideal M.

Proof. Let $a/s, b/t \in V$, where $a, b \in R$ and $s, t \in S$. Then $at, bs \in R$. We show that atV and bsV are comparable, and hence $(a/s)V$ and $(b/t)V$ are comparable. Thus we need only show that aV and bV are comparable for $a, b \in R$. In fact, we may assume that $a, b \in M$.

First, aM and bR are comparable since R is a PVR. If $bR \subset aM$, then $b = an$ for some $n \in M$, and hence $bV \subset aV$. Thus we may assume that aM is properly contained in bR. Let $s \in M$ be a non-zerodivisor. Then $as = br$ for some $r \in R$. Now rM and sR are comparable since R is a PVR. If $sR \subset rM$, then $s = rm$ for some $m \in M$. Then r and m are not zerodivisors since s is not a zerodivisor. Thus, since $as = br$, we have $arm = br$, and $am = b$ since r is not a zero-divisor. Thus $bV \subset aV$. We may thus assume that rM is properly contained in sR. Hence $rM \subset sM$, so $(r/s)M \subset M$, and thus $r/s \in (M : M) = V$. Since $as = br$, $a = b(r/s)$ with $r/s \in V$. Hence $aV \subset bV$, and V is a chained ring.

Finally, we show that M is the maximal ideal of V, that is, each $x \in V - M$ is a unit of V. Let $x = r/s \in V - M$, where $x \in R$ and $s \in S$. If $s \in R - M$, then s is a unit of R. Thus $x = rs^{-1} \in R - M$, and hence x is a unit in R, and thus a unit in V. Hence we may assume that $s \in M$. Thus sM and rR are comparable since R is a PVR. If $rR \subset sM$, then $r = sm$ for some $m \in M$, and hence $x = r/s = sm/s = m \in M$, a contradiction. Thus sM is properly contained in rR. Hence $s^2 = rt$ for some $t \in R$. Thus both r and t are non-zero-divisors since s is a non-zerodivisor. Hence $x^{-1} = s/r \in R_S$. Since sM is properly contained in rR, $sM \subset rM$, and hence $(s/r)M \subset M$. Thus $x^{-1} = s/r \in (M : M) = V$. Hence M is the unique maximal ideal of V. □

Corollary 9. Let R be a PVR whose maximal ideal M contains a non-zerodivisor. Then R is the pullback $V \times_F k$, where $V = (M : M)$ is a chained ring and $k = R/M \subset V/M = F$. □

Together with Theorem 6, Corollary 9 recovers the pullback characterization of PVDs mentioned in the Introduction. However, our next example shows that the hypothesis in Theorem 8 and Corollary 9 that M contains a non-zerodivisor is necessary.

Example 10. **(a)** Let k be a field and X and Y indeterminates. Then $R = k[X, Y]/(X^2, XY, Y^2) = k[x, y]$ is quasilocal with maximal ideal $M = (x, y)$. Note that $zM = 0$ for any $z \in M$, and hence R is a PVR with $Z(R) = M$. However, $R = (M : M)$, and R is not a chained ring since the ideals xR and yR are not comparable.

(b) A one-dimensional PVR, R, with maximal ideal M which consists only of zerodivisors. Let $V = \mathbb{Z}_{(2)} + X\mathbb{Q}[[X]]$, a two-dimensional valuation domain with prime ideals $0 \subset P = X\mathbb{Q}[[X]] \subset M = 2\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]]$. Let $R = V/X^2V$. Then R is a PVR by Corollary 3, and $\dim R = 1$. Note that $\mathrm{nil}(R) = P/X^2V$ (nil(R) is the set of nilpotent elements of R), and $Z(R) = M/X^2V$ since $(x^2/2)M \subset X^2V$, but $X^2/2 \notin X^2V$ since $1/2 \notin V$. □

We have the following converse to Theorem 8.

Theorem 11. Let R be a quasilocal ring with maximal ideal M. If $V = (M : M)$ is a chained ring with maximal ideal M, then R is a PVR.

Proof. By Corollary 4, V is a PVR. Hence R is a PVR by Theorem 7. □

We next show that any proper localization of a PVR is a chained ring (cf. [8, Proposition 2.6]). (Note that any proper localization of a PVR, R, has the form R_P for a nonmaximal prime ideal P of R since Spec(R) is linearly ordered. In particular, R_S, the total quotient ring of a PVR, R, is a chained ring if $R_S \neq R$.)

Theorem 12. Let R be a PVR with maximal ideal M and P a nonmaximal prime ideal of R. Then R_P is a chained ring. Moreover, $R_P = (M : M)_P$.

Proof. Let $x, y \in R_P$. We need only consider the case when $x = a/1$ and $y = b/1$ for some $a, b \in P$. Suppose, without loss of generality, that a does not divide b in R. Thus $aM \subset bR$ since R is a PVR. Since R is quasilocal and P is not maximal, there is an $s \in M - P$. Thus $as = br$ for some $r \in R$.

Hence $a = b(r/s)$ with $r/s \in R_P$, and so $aR_P \subset bR_P$. Thus R_P is a chained ring.

For the "moreover" statement, choose $t \in M - P$, and let $x \in (M : M)$ and $s \in R - P$. Then $x/s = xt/(st) \in R_P$. Hence $(M : M)_P \subset R_P$, and thus $R_P = (M : M)_P$. □

The "moreover" statement in Theorem 12 also follows by combining Corollary 9 with general results about pullbacks.

We say that an ideal I of R satisfies property (*) if whenever $xy \in I$ for $x, y \in R_S$, then either $x \in I$ or $y \in I$. Such an ideal I is necessarily prime, and in the domain case, I is strongly prime. If R is a von Neumann regular ring, or more generally any ring R with $R = R_S$, then all maximal ideals of R satisfy (*) and R need not be quasilocal. However, we do show that a maximal ideal which is strongly prime satisfies (*). First a lemma.

Lemma 13. Let R be a PVR with maximal ideal M. If $y \in R_S - R$, then $y^{-1} \in (M : M)$.

Proof. Since $R_S \neq R$, there is a non-zerodivisor $z \in M$. Write $y = r/s$ with $r \in R$ and $s \in S$. Then rR and sM are comparable since R is a PVR. Since $r/s \notin R$, also $r \notin sM$, and hence sM is properly contained in rR. Thus $sM \subset rM$. Hence $sz = rt$ for some $t \in M$. Note that r and t are not zerodivisors since s and z are not zerodivisors. Thus $y^{-1} = s/r \in R_S$, and hence $(s/r)M \subset M$; so $y^{-1} \in (M : M)$. □

Theorem 14. Let R be a PVR with maximal ideal M. Then M satisfies (*).

Proof. Let $xy \in M$ with $x, y \in R_S$. If $x, y \in R$, then either $x \in M$ or $y \in M$. If $x \in R_S - R$, then $x^{-1} \in (M : M)$ by Lemma 13. Hence $y = x^{-1}(xy) \in M$. □

Remark 15. The converse to Theorem 14 is false; just let R be any von Neumann regular ring which is not quasilocal. For a quasilocal example, let k be a field, X, Y, and Z be indeterminates, and $R = k[X, Y, Z]/(X^2, Y^2, Z^2) = k[x, y, z]$.

Then R is quasilocal with maximal ideal $M = (x, y, z)$ and $Z(R) = M$; so $R_S = R$, and hence M satisfies (*). However, R is not a PVR since xM and yR are not comparable because $xz \notin yR$ and $y \notin xM$. □

We next determine necessary and sufficient conditions for a maximal ideal which satisfies (*) to be strongly prime. Of course, the concepts coincide when R is an integral domain.

Theorem 16. Let R be a quasilocal ring whose maximal ideal M satisfies (*). Then M is strongly prime if and only if

(1) Z(R) is a prime ideal of R and is comparable to each principal ideal of R, and

(2) xM and yR are comparable for all $x, y \in Z(R)$.

Proof. Suppose M is strongly prime, and hence (2) holds. Then R is a PVR by Theorem 2. By Lemma 1(a), Spec(R) is linearly ordered by inclusion; so, Z(R) is a prime ideal (cf. [10, p. 3 and Theorem 9]). Hence by Lemma 1(a), (1) also holds.

Conversely, suppose that R satisfies (1) and (2). We show that xM and yR are comparable for all $x, y \in M$. If $x, y \in Z(R)$, this follows from (2). If $x \in Z(R)$ and $y \notin Z(R)$, then by (1), $xR \subset Z(R) \subset yR$; hence $xM \subset yR$. If $x \notin Z(R)$ and $y \in Z(R)$, then $yR \subset Z(R)$; also, $Z(R) \subset xM$ since if $xM \subset Z(R)$, then $x^2 \in Z(R)$, a contradiction since Z(R) is a prime ideal of R. Finally, suppose that $x \notin Z(R)$, $y \notin Z(R)$, and $xM \not\subset yR$. Then $(x/y)M \not\subset R$, and hence $xm/y \notin R$ for some $m \in M$. Then $(y/x)(xm/y) = m \in M$, so $y/x \in M$ since M satisfies (*). Hence $yR \subset xM$, and so M is strongly prime. □

We next study the integral closure R' of a PVR, (R, M). We first show that $R' \subset (M : M)$ in general.

Lemma 17. Let R be a PVR with maximal ideal M. Then $R' \subset (M : M)$.

Proof. Let $x = r/s \in R'$, where $r \in R$ and $s \in S$. Then sR and rM are comparable since R is a PVR. If $sR \subset rM$, then $s = rm$ for some $m \in M$. Thus m is not a zerodivisor; so $1/m = r/s \in R'$. But then 1/m integral over R implies that m is a unit of R, a contradiction. Thus $rM \subset sM$, that is, $(r/s)M \subset M$. Hence $R' \subset (M : M)$. □

Our next example shows that the containment in Lemma 17 may be strict even in the integral domain case. Theorem 21 gives necessary and sufficient conditions for $R' = (M : M)$.

Example 18. Let k be a field, X and Y indeterminates, and $R = k + Yk(X)[[Y]]$. Then R is an integrally closed PVD (cf. [8, Example 2.1]) which is properly contained in its associated valuation overring $k(X)[[Y]] = (M : M)$, where $M = Yk(X)[[Y]]$. □

Theorem 19. Let R be a PVR with maximal ideal M. Then R' is a PVR with maximal ideal M.
Proof. By Lemma 17, $R' \subset (M : M)$; hence M is a prime ideal of R' by Theorem 14. By integrality, M is the unique maximal ideal of R'. By Theorem 7, R' is a PVR. □

We will need Lemma 13 and the following lemma for the proof of Theorem 21.

Lemma 20. Let R be a PVR with maximal ideal M. Let B be an overring of R. If $s^{-1} \in B$ for some non-zerodivisor $s \in M$, then B is a chained ring.
Proof. Let $y \in B - R$. Then $y = a/t$, where $a \in R$ and $t \in S$. Since aR and tM are comparable, necessarily $tM \subset aR$ since $a/t \notin R$. Thus $st = ar$ for some $r \in R$. Hence $a \in S$. Thus $y^{-1} = t/a = r/s \in B$ since $s^{-1} \in B$. Hence $B - R \subset U(B)$, the set of units of B. Thus to show that B is a chained ring, we need only show that aB and bB are comparable for $a, b \in R$. Since R is a PVR, aM and bR are comparable. If $aM \subset bR$, then $as = bx$ for some $x \in R$, and hence $a = b(x/s)$ gives $aB \subset bB$. If $bR \subset aM$, then $bB \subset aB$. Hence B is a chained ring. □

Our final result generalizes a result for integral domains ([9, Proposition 2.7], [6, Proposition 4.2]).

Theorem 21. Let R be a PVR with maximal ideal M. Then $R' = (M : M)$ if and only if every overring of R is a PVR.
Proof. Suppose that $R' = (M : M)$. Let B be an overring of R. By Lemma 20 and Corollary 4, we may assume that $s^{-1} \notin B$ for all non-zerodivisors $s \in M$. Let $y = r/s \in B - R$, where $r \in R$ and $s \in S$. By Lemma 13, $y^{-1} \in (M : M)$, and hence $y^{-1} \in R'$. Thus $y^{-1} \in R[y] \subset B$; so y is a unit of B. Also, $y^{-1} \notin R$ since otherwise $t = y^{-1} \in M$ would be a non-zerodivisor with $t^{-1} \in B$. Thus $y \in B - R$ implies $y^{-1} \in B - R$, and hence $y = (y^{-1})^{-1} \in (M : M) = R'$ by Lemma 13. Thus M is the unique maximal ideal of B. Hence B is a PVR by Theorem 7.

Conversely, suppose R' is properly contained in $(M : M)$. Then $R'/M \subset (M : M)/M$ is not an algebraic extension. Hence there is an intermediate ring which is not quasilocal. Its inverse image yields a nonquasilocal ring A between R' and $(M : M)$ (cf. [3, Corollary 3.26]). By Lemma 1(b), A is not a PVR.

REFERENCES

1. D. F. Anderson, Comparability of ideals and valuation overrings, *Houston J. Math.* **5**(1979), 451-463.
2. D. F. Anderson, When the dual of an ideal is a ring, *Houston J. Math.* **9**(1983), 325-332.
3. D. F. Anderson and D. E. Dobbs, Pairs of rings with the same prime ideals, *Can. J. Math.* **32**(1980), 362-384.
4. A. Badawi, A visit to valuation and pseudo-valuation domains, pp. 155-161, in *Commutative Ring Theory*, Lecture Notes Pure Appl. Math., vol. 171, Marcel Dekker, Inc., New York / Basel, 1995.
5. A. Badawi, On domains which have prime ideals that are linearly ordered, *Comm. Algebra* **23**(1995), 4365-4373.
6. D. E. Dobbs, Coherence, ascent of going-down and pseudo-valuation domains, *Houston J. Math.* **4**(1978), 551-567.
7. D. E. Dobbs, M. Fontana, J. A. Huckaba and I. J. Papick, Strong ring extensions and pseudo-valuation domains, *Houston J. Math.* **8**(1982), 167-184.
8. J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, *Pac. J. Math.* **75**(1978), 137-147.
9. J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, II, *Houston J. Math.* **4**(1978), 199-207.
10. I. Kaplansky, *Commutative Rings*, rev. ed., Univ. Chicago Press, Chicago, 1974.